2) the sum of the mean function and the amplitude with phase function. However, if the physics of the problem of interest is understood so that modeling one or both of the smooth functions with as few coefficients is possible, a generalization of Ksienski's algorithm using least squares to minimize with respect to the known functions could be much more accurate and effective.

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Comments on "Fast Interpolation Algorithm Using Fast Hartley Transform"

CHAU-YUN HSU, TENG-PIN LIN, AND
JA-LING WU

Certain errors in the aboved titled letter[1] *are pointed out and corrected.*

In the above letter,[1] Agbinya proposed a fast interpolation algorithm using fast Hartley transform which reduces the computational load and the mean-square error as compared with the FFT approach. But unfortunately, there are several errors in the original letter as indicated in the following.

The gain of the low-pass filter for the interpolation $H(mT_f)$ should be

$$H(mT_f) = \text{rect}\,(W) = \begin{cases} S + 1, & |W| < \pi/T \\ 0, & \pi/T < |W| < \pi/T_f. \end{cases}$$

Further, (8) should be also corrected as

$$\begin{aligned} Z_k &= (S + 1)\, F_k, & k &= 0, 1, 2, \cdots, (N/2) - 1 \\ Z_k &= 0, & k &= N/2, (N/2) + 1, \cdots, P - (N/2) - 1 \\ Z_k &= (S + 1)\, F_k, & k &= P - (N/2), P - (N/2) + 1, \cdots, P - 1. \end{aligned}$$

The original (8) is used for decimation not for interpolation.

And the desired points $Z(m)$ of the new sequences are obtained by getting the inverse DHT of (8) as

$$Z(M) = 1/P \sum_{k=0}^{P-1} Z_k \,\text{cas}\,(b/P), \qquad m = 0 \text{ to } P - 1.$$

That is, the transform kernal of (9) should be cas (b/P) instead of cas (b/N) as described in the original letter, where $P = N + S^*(N - 1)$. Therefore, the complexity analysis is questionable, for the misusing of transform kernal.

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[1] J. I. Agbinya, "Fast interpolation algorithm using fast Hartley transform," *Proc. IEEE*, vol. 75, no. 4, pp. 523-524, Apr. 1987.

Author's Reply[2]

The suggested corrections [1] to the gain of the low-pass filter and the kernel of the cas function (9) respectively of [2] are in order. However, Hsu *et al.* have introduced a further error into (9) by using $Z(M)$ instead of $z(m)$. The correct version is

$$z(m) = 1/P \sum_{k=0}^{P-1} Z_k \cdot \text{cas}\,(b/P), \qquad m = 0 \text{ to } P - 1.$$

Their form of (8) is usually used for decimation and leads to large errors, more noticeable when N is small in both FFT and DHT applications. The use of (8) of [2] in (9) lessens serious errors and requires restoring scaling factors to the amplitudes of $z(m)$.

The complexity analysis expressions in [2] are not dependent on P as Hsu *et al.* have implied in [1], since no additions and multiplications need to be done for the trivial cases of $Z_k = 0$ in the range $k = N/2$ to $P - (N/2) - 1$ in both (8) and (9). By such considerations, it is infact possible to further reduce both addition and multiplication counts respectively down to

$$\text{Addition} = 2N - 2 + (2N - 4)\ \text{Int} \cdot (\log_4 N)$$

and

$$\text{Multiplication} = 1 + 2 \cdot N + (N - 4) \cdot \text{Int}\,(\log_4 N).$$

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[2] Manuscript received October 19, 1987.

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On the Number of Costas Arrays as a Function of Array Size

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A conjecture that the number of Costas arrays is a monotonic increasing function of array size is disproved by extension of the known values to n = 17. A probabilistic estimation formula is developed which predicts the peak at n = 16 and tracks the known values to (typically) 5-6 percent.

I. INTRODUCTION

A Costas array or "constellation" [1] is a pattern of n dots on an $n \times n$ grid, one dot per row and one per column, in which the $n(n - 1)/2$ vectors between the dots are all distinct. Such patterns, first identified by J. P. Costas, provide a template for generating radar and sonar signals with ideal ambiguity functions [2]-[4] and also have potential utility for problems in physical alignment and synchronization.

While no construction for generating a Costas array valid for general n has been found, Golomb and Taylor [5] present a variety of constructions for special n values based on the properties of primitive roots of finite fields. In a list of open questions on the number of Costas array $C(n)$, they include the conjectures: 1) $C(n) \geq 1$ for all $n \geq 1$, and 2) $C(n)$ is monotonic increasing. We report here a further extension to $n = 17$ of the known values of $C(n)$ [6]. Our results disprove the second conjecture and cast doubt on the first. Further, we derive a simple probabilistic estimation formula wtih one free parameter which predicts the peak in $C(n)$ at $n = 16$ and tracks the known values of $C(n)$, which vary over four orders of magnitude, to (typically) 5-6 percent.

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II. PROBABILITY MODEL

A Costas array can alternatively be defined as the subset of the $n!$ permutations of 1 to n which have no repeated entries (corresponding to no repeated vectors) on any line of the difference triangle (DT) associated with the permutation. Fig. 1 shows a Costas array for $n = 6$ with its associated permutation and DT. The first line of the DT is generated by differences between adjacent elements in the permutation, the second line by second neighbor differences, etc. If P_{NR} is the probability for a given n that the DT associated with a random permutation has no repeats, then

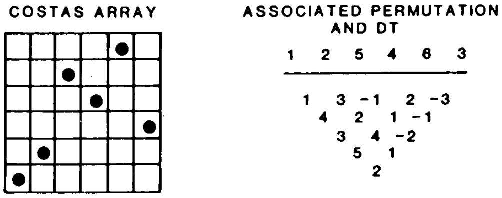

Fig. 1. The 6 by 6 Costas array associated with the permutation: 1 2 5 4 6 3.

$$C(n) = n!\,(P_{NR}). \tag{1}$$

We began by simply computing the probability for a given n that each line of the DT has no repeats assuming the allowed values ($\pm 1, \pm 2, \cdots, \pm(n-1)$) are equally probable and never depleted. Ignoring any correlation between the lines of the DT, one then multiplies each of these line probabilities together to estimate P_{NR}. The very crude estimates of $C(n)$ then obtained from (1) reflected generally the trend in the data (then known only to $n = 14$) and predicted a maximum at $n = 15$ or 16 (see Table 1, column labeled "crude estimate").

Table 1 Comparison of Exact $C(n)$ with Estimates

n	IP	Exact $C(n)$	Crude Estimate	$C(n)$ from (6)	Gaussian Fit	K
1	0	1	—	1	0	—
2	0	2	—	2	0	—
3	1	4	—	4.3	0.2	1.33
4	3	12	11	11.3	1.3	1.03
5	7	40	28	29	6.5	0.87
6	13	116	71	77	28	0.92
7	22	200	170	191	101	1.091
8	34	444	390	467	322	1.118
9	50	760	829	1035	890	1.161
10	70	2160	1606	2183	2132	1.107
11	95	4368	2819	4098	4436	1.099
12	125	7852	4449	7152	8011	1.097
13	161	12828	6277	10996	12557	1.093
14	203	17252	7860	15459	17084	1.098
15	252	19612	8697	18997	20174	1.104
16	308	21104	8668	21085	20677	1.106
17	372	18276	7199	20334	18395	1.111

Then we refined the model to simulate two features which characterize the DTs associated with permutations: 1) the correct frequency of allowed values, namely, $(n-1)$ entries of ± 1, $(n-2)$ entries of $\pm 2, \cdots$, one entry of $\pm(n-1)$ (note the sample DT in Fig. 1; and 2) the correlation between lines of the DT. Repeats typically occur in two related pairs on separate lines of the DT (whenever four elements of the permutation are involved in the repeat). For example 1 5 4 2 6 3 will generate a related repeat on the first and third line of the DT. The following Lemma summarizes this correlation. If line one of a DT is free of repeats, line 2 can have no adjacent repeats; if lines 1 and 2 are free of repeats, any three adjoining entries of line 3 must be different, etc. (This observation is equivalent to Chang's weakened definition of a Costas array [7].) We incorporate feature 2 into the model through the distinction between the total number of pairs of elements added line by line on the DT, $TP(n)$ where

$$\begin{aligned} TP(n) &= (n-1)(n-2)/2 + (n-2)(n-3)/2 + \cdots + 1 \\ &= n(n-1)(n-2)/6 \end{aligned} \tag{2}$$

and the number of independent (or more precisely, not known to be correlated) pairs line by line on the DT, $IP(n)$, where as a consequence of the Lemma above

$$\begin{aligned} IP(n) &= (n-1)(n-2)/2 + (n-3)(n-4)/2 + \cdots + (1 \text{ or } 3) \\ &= (n)(n-2)(2n+1)/24 \qquad (n \text{ even}) \\ &= (n+1)(n-1)(2n-3)/24 \qquad (n \text{ odd}). \end{aligned} \tag{3}$$

We next postulate that the number of Costas arrays will follow an expression of the form:

$$C(n) \approx n!\,(1 - \bar{P}_R(n))^{IP(n)} \tag{4}$$

where $\bar{P}_R(n)$ is viewed as a representative average (in the sense of a geometric mean) probability per independent pair of a repeat in the DT. To test the validity of this assumption, we need to predict the n-dependence of $\bar{P}_R(n)$. Taking $\bar{P}_R(n)$ as the sum of mutually exclusive probabilities that a ± 1 repeats, that a ± 2 repeats, $\cdots$ up to that a $\pm(n-2)$ repeats and designating $P(n)$ as the last (smallest) of these probabilities, we can write in view of the frequency of entries on the DT that

$$\begin{aligned} \bar{P}_R(n) &\propto [(n-1)(n-2) + (n-2)(n-3) + \cdots + 2 \times 1]\, P(n) \\ &\propto TP(n)\, P(n). \end{aligned} \tag{5}$$

Further, the "baseline" probability $P(n)$ should vary inversely as $(T)(T-1)$ where T is the total number of entries on the DT, i.e., $(n)(n-1)/2$. Hence

$$\bar{P}_R(n) \propto TP(n)/[(n)(n-1)/2]\,[(n)(n-1)/2 - 1]$$

which combined with (2) reduces to

$$\bar{P}_R(n) \propto 2/3(n+1).$$

Thus (4) is recast to

$$C(n) \approx n![1 - K/(n+1)]^{IP(n)} \tag{6}$$

where K is an unknown constant of proportionality.

III. COMPARISON TO EXACT RESULTS

The initial success in the correlation between (6) and the known $C(n)$ values, and the doubt cast on conjecture 2 [6], led us to extend the exact computation of $C(n)$ to $n = 17$. (Identical values to ours through $n = 15$ have been computed independently by Robbins and Taylor [8].) A series of nine runs in a low priority background mode on three Sun workstations produced the $C(17)$ value.

Table 1 lists all the known exact values and several sets of estimates. The values from (6) based on a least-squares optimized value of $K = 1.105787$ fit the exact results from $n = 1$ to $n = 17$ to an average of 10.4 percent. The values of K for each n which reproduce the exact $C(n)$ are also given in Table 1.

Fig. 2 shows the exact values, the fit with (6) converted to continuous form by Stirling's approximation to the factorial function, and an optimized three-parameter Gaussian curve fit to the exact values inspired by the Gaussian look of the data. The latter fits the exact values somewhat better near the peak, but the data for smaller n are more consistent with (6).

IV. FINAL COMMENTS AND CONCLUSIONS

A word is in order on the limitations of the present model. First, we are estimating a complex conditional probability by means of one representative geometric mean probability. Second, we are assuming that the set of random arrangements of the DTs which we model are statistically similar to the $n!$ subset generated from "master" permutations. Finally, in treating the fraction of permutations which are Costas arrays as a random phenomenon, we are ignoring the perturbing effects of nonrandom constructions for specific n values (in fact, our results suggest that $C(n)$ would ultimately be dominated by such constructions).

Despite these caveats, (6) does remarkably well in tracking the exact values. Chronologically, the peak in $C(n)$ was predicted and afterwards confirmed. Note (see Fig. 2) that the predicted random values for $C(n)$ are much less than unity for $n \geq 30$; contrary to conjecture 1, could the exact values be zero for some of these n? Observing that the first gaps in the list of known constructions [5,

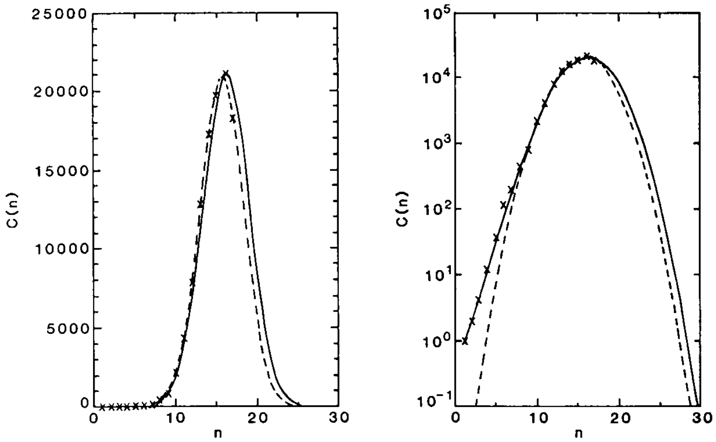

Fig. 2. The exact values of $C(n)$ compared to the predictions of (6) (solid line) and to the Gaussian approximation, $20833 \exp[-0.07080(n - 15.6734)^2]$, (dashed line). Results are shown on linear (left) and semilog (right) scales.

Fig. 10] occur at $n = 32$ and $n = 33$, we challenge the reader to find a Costas array for either of these n.

ACKNOWLEDGMENT

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