Interlaced Costas arrays do not exist

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Abstract

We prove that the only Costas arrays that can be constructed by interlacing 2 Costas arrays of smaller orders (either equal or differing by 1) are those of order 2, and that, consequently, no non-trivial Costas arrays result from this method.

1 Introduction

The longest-standing open problems in the field of Costas arrays [1, 2] are, at the same time, the most fundamental ones: the existence of Costas arrays for all orders, and techniques for their construction [3, 6, 7]. In an effort to construct new Costas arrays (where by "new" we mean arrays that neither have been discovered yet through exhaustive search nor can be constructed by the known algebraic techniques [3, 6, 7]), one is tempted to apply various empirical or semi-empirical improvised methods that may lead to a Costas array. One of the first ideas that spring into mind is to "interlace" 2 known Costas arrays into a larger one, hoping that the result will be a (new) Costas array. By "interlacing" we mean that the new array will contain columns alternatively chosen from the 2 Costas arrays, so that all odd columns come from the first and all even from the second, and similarly for the rows (see Figure 1). It looks at first that such a technique is promising, as it mixes the arrays while preserving the Costas property for each one of them.

It is clear that interlacing can be attempted with arrays of either the same order (for example, 2 Costas arrays of order 16 can be combined into an array of order 32), or of orders differing by 1 (for example, a Costas array of order 17 and a Costas array of order 16 can be combined into an array of order 33). Virtually every novice in the field spends several hours playing around with this idea, until (s)he gets eventually disappointed as no array produced seems to have the Costas property. In this work we prove that neither case of interlacing can produce a Costas array, except perhaps in trivial situations where the interlaced arrays are very small. Although the proofs we offer are quite elementary and straightforward, they are significant in the sense that they rely exclusively on the Costas property, and not on any extra assumptions. Indeed, it is notoriously difficult to prove rigorously results for Costas arrays in general, and most rigorous results available are about the algebraically constructed Costas arrays [6, 7, 4]. A proof on the futility of interlacing can save researchers in the field precious time.

2 Basics

For reasons of completeness, we give below some basic definitions about Costas arrays [3].

Definition 1. Let $n \in \mathbb{N}$, $[n] = \{1, \dots, n\}$, $a : [n] \to [n]$ be a bijection, namely a permutation on the integers $1, \dots, n$; its *difference triangle* $T(a)$ is defined to be the collection of the multisets $\{t_i(a) : i = 1, \dots, n-1\}$, where $t_i(a) = \{a(i+j) - a(j) : j = 1, \dots, n-i\}$ (it is customary to call t_i the ith *row* of the triangle). Its corresponding array is $A = [\alpha_{ij}],\ i, j \in [n]$, so that

$$\alpha_{ij} = \begin{cases} 1, & a(j) = i \\ 0, & \text{otherwise} \end{cases}$$

and it is customary to denote 1s by dots and 0s by blanks. The collection of vectors $\{(i-j, a(i) - a(j)) : i > j, i, j \in [n]\}$ are the *distance vectors* of A (or a).

Definition 2. Let $a : [n] \to [n]$ be a bijection; it will have the *Costas property* iff the multisets in $T(a)$ are

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$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4)\\ (2,1) & (2,2) & (2,3) & (2,4)\\ (3,1) & (3,2) & (3,3) & (3,4)\\ (4,1) & (4,2) & (4,3) & (4,4) \end{bmatrix}, \begin{bmatrix} \mathbf{(1,1)} & \mathbf{(1,2)} & \mathbf{(1,3)} & \mathbf{(1,4)}\\ \mathbf{(2,1)} & \mathbf{(2,2)} & \mathbf{(2,3)} & \mathbf{(2,4)}\\ \mathbf{(3,1)} & \mathbf{(3,2)} & \mathbf{(3,3)} & \mathbf{(3,4)}\\ \mathbf{(4,1)} & \mathbf{(4,2)} & \mathbf{(4,3)} & \mathbf{(4,4)} \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} (1,1) & & (1,2) & & (1,3) & & (1,4) & \\ & \mathbf{(1,1)} & & \mathbf{(1,2)} & & \mathbf{(1,3)} & & \mathbf{(1,4)}\\ (2,1) & & (2,2) & & (2,3) & & (2,4) & \\ & \mathbf{(2,1)} & & \mathbf{(2,2)} & & \mathbf{(2,3)} & & \mathbf{(2,4)}\\ (3,1) & & (3,2) & & (3,3) & & (3,4) & \\ & \mathbf{(3,1)} & & \mathbf{(3,2)} & & \mathbf{(3,3)} & & \mathbf{(3,4)}\\ (4,1) & & (4,2) & & (4,3) & & (4,4) & \\ & \mathbf{(4,1)} & & \mathbf{(4,2)} & & \mathbf{(4,3)} & & \mathbf{(4,4)} \end{bmatrix}$$

Figure 1: A demonstration of interlacing for 2 4×4 arrays: the elements of the first are in normal and of the second in **bold** script; the interlaced array has rows and columns alternatively chosen from each of the arrays. As a result, the elements of the 2 old arrays occupy only those elements of the new array whose coordinates have the same parity modulo 2 (are either both even or both odd).

actually sets, namely none of them contains a repeated entry, or, equivalently, when the distance vectors of a are all distinct.

Definition 3. Let $a : [n] \to [n]$, $b : [m] \to [m]$, $m \leq n$ be Costas permutations; they will be *orthogonal* iff the multisets $t_i(a) \cup t_i(b),\ i \in [m-1]$ are actually sets, or, equivalently, when the distance vectors of a and the distance vectors of b have no vector in common.

The Costas property can be reformulated in terms of the auto-correlation of Costas arrays/permutations:

Definition 4. Let $a, b : [n] \to [n]$, $n \in \mathbb{N}$, and let $u, v \in \mathbb{Z}$; the cross-correlation between a and b at (u, v) is defined as

$$\Psi_{a,b}(u,v) = |\{(a(i)+v, i+u) : i \in [n]\} \cap \{(b(i), i) : i \in [n]\}|.$$

When $a = b$, $\Psi_{a,a}$ is the auto-correlation of a. A local maximum $M = (u_0, v_0)$ of $|\Psi|$ along with the largest neighborhood S of M such that M is the only local maximum of $|\Psi|$ in S is called a *lobe* of Ψ: the *main lobe* corresponds to the global maximum (note that it may not be unique in case of multiple equal global maxima), and the *sidelobes* to the remaining maxima.

Geometrically, the cross-correlation is interpreted as follows: place the arrays A and B, corresponding to a and b respectively, on top of each other, so that they overlap perfectly, then slide A by u columns to the right and v rows downwards, and set $\Psi_{a,b}(u,v)$ to be the number of pairs of overlapping dots.

In the next section Costas arrays will be interpreted as signal representations in the time-frequency plane (horizontal and vertical directions, respectively), and a dot will denote the presence of energy in a certain area of the time-frequency plane; we assume that the array tiles the plane into equal areas, and that all dotted areas carry the same amount of energy. In this context, the cross-correlation, as defined above, is based on the energy content alone (and disregards completely phase information).

Finally, we should also define interlacing more formally (see Figure 1 for an example of interlacing of 2 4×4 arrays):

Definition 5. Let $a : [n+s] \to [n+s]$ and $b : [n] \to [n]$ be permutations, and let $s \in \{0, 1\}$; then, their interlaced permutation is $c : [2n+s] \to [2n+s]$, where

$$c(i) = \begin{cases} 2a\left(\frac{i+1}{2}\right) - 1, & i \equiv 1 \mod 2 \\ 2b\left(\frac{i}{2}\right), & i \equiv 0 \mod 2 \end{cases}, i \in [2n+s]$$

It is clear that, within the array corresponding to c, the distance vectors of a and b are still present, albeit dilated by a factor of 2. If then a and b are non-orthogonal Costas permutations, c cannot be Costas, because its corresponding array will have a pair of equal distance vectors; the same is obviously true if either a or b fail to be Costas. Therefore, our only hope for interlacing to yield a Costas array is to start with Costas arrays (of equal orders or orders differing by 1). Note that a feature of such interlaced Costas arrays would be that their dots lie exclusively at positions with coordinates either both even or both odd, and this may already sound alarmingly restrictive; indeed, as we are about to show, such Costas arrays do not exist, except in trivial cases.

3 Why Costas arrays?

RADARs and SONARs detect the the distance and velocity of targets around them by transmitting periodically a waveform W and listening for reflections R. Assuming an ideal noiseless environment, R is just a copy of W, only attenuated, and shifted in frequency and time. The time delay indicates the distance of the target, while the frequency shift, through the Doppler effect, its velocity (we assume here that the frequency content of W is narrowband enough for the Doppler effect, which is multiplicative, to be well approximated by a uniform additive shift for all frequencies).

How are the time and frequency shifts detected? The simplest solution would be to apply a matched filter: R is cross-correlated with shifted versions of W for various time and frequency shifts, and the pair of shifts corresponding to the maximal cross-correlation are the true shifts sought. Alas, this simple idea fails to work in practice, because all real media are incoherent: phase delay varies with frequency, hence waveforms tend to spread while traveling in the medium, so that, by the time R reaches the RADAR/SONAR, it looks nothing like W any more.

J. P. Costas's idea [2] was to discard phase information, since it is unreliable, and carry out the cross-correlation based on the energy contents of W and R alone. Consider a waveform of the form:

$$W(t) = A\cos\left(\phi_k + 2\pi\left(f_0 + \frac{a_k}{n}f_1\right)t\right),\quad t\in\left[\frac{k-1}{n}T,\frac{k}{n}T\right],$$

where $k\in[n]$, T is the time duration of the pulse, f_0, f_1 are two predetermined frequencies, ϕ_k are phases suitably chosen so that the phase of W is continuous in t (we may choose $\phi_1 = 0$), and $a:[n]\to[n]$ is a bijection. This is a frequency hopping waveform whose instantaneous frequency is

$$f(t) = f_0 + \frac{a_k}{n}f_1,\ t\in\left(\frac{k-1}{n}T,\frac{k}{n}T\right), k\in[n]$$

We observe that W is completely determined by a, given that f_0, f_1, and T are set.

Costas's idea amounts effectively to placing an energy content detector before the matched filter, thus reconstructing a from W, and similarly for R: the signals fed to the matched filter can then each be abstracted as a 2D infinite sequence, representing the time-frequency plane: this sequence is full of 0s/blanks (energy is not present), except for a $n\times n$ square than corresponds to a permutation array (whose 1s/dots denote that energy is present there), exactly as described in Definition 1. The filter overlays the two 2D sequences, then shifts one with respect to the other by some rows vertically and some columns horizontally, and counts how many pairs of dots overlap:

$$\Psi_{A,B}(u,v) = \sum_{i,j} a_{ij}b_{i+u,j+v},$$

where Ψ is the cross-correlation, $A,B:\mathbb{Z}^2\to\{0,1\}$ the two 2D sequences, and u, v the shift parameters. In the absence of noise, R is an exact copy of W, only shifted in time and frequency, so the matched filter will have found the correct shift parameters when the cross-correlation becomes equal to n.

When noise is present, however, some of R's dots may have shifted irregularly or even gone altogether missing: R will no longer be an exact copy of W, and the maximal cross-correlation will no longer be n. The filter will have no alternative than to locate the maximal cross-correlation (note that now it won't know a priori what the maximum will be) and return the shift parameters corresponding to it; but this maximum may no longer be unique, or one of the (former) sidelobes may have grown taller than the main lobe: either case will result to spurious target detection.

What should the form of a (or the corresponding array A) be in order to minimize the probability of spurious detections? In the absence of noise, the cross-correlation is just a shifted form of the autocorrelation of A, so we need to choose A in such a way as to suppress as much as possible the height of the autocorrelation sidelobes relatively to the main lobe (whose height is n):

$$A = A^* = \operatorname*{argmin}_A \max_{(u,v)} \Psi_{A,A}(u,v).$$

Choosing any pair of dots in A, there exists a shift (their distance vector) that will move these dots on top of each other, so sidelobes of height 1 will exist and nothing can be done about it. If, however, we stipulate that distance vectors be unique, there will be no sidelobe of height 2 or more; but this is precisely the Costas property (see Definitions 1 and 2)! Autocorrelation is known as *auto-ambiguity* in the SONAR/RADAR community, and waveforms with the Costas property are said to have *ideal thumbtack auto-ambiguity* [2, 7].

Why should the optimal A be a permutation array, as we assumed (summarily and without any further explanation) above? Would using twice the same frequency, or using two frequencies simultaneously, not improve the autocorrelation? Costas argued on basic engineering principles that indeed it would not [2].

4 The quest for orthogonal Costas arrays

The entries of the difference triangle of a permutation exhibit strong dependence; so strong, in fact, that even the 3 entries at its bottom (namely in rows $n-1$ and $n-2$, $n > 2$ being the order of the permutation) are sometimes enough to guarantee that a (Costas) permutation with the given difference triangle cannot possibly exist!

Lemma 1. *Let $a : [n] \to [n]$ be a function, and let $t_{n-1}(a) = \{x\}$, $t_{n-2}(a) = \{y, z\}$, $n, x, y, z \in \mathbb{N}$; then,*

- *If $x = y$ or $x = z$, a cannot be a permutation;*
- *If $x = y + z$, a cannot be a permutation, unless perhaps $n = 3$.*

Proof. $t_{n-1}(a) = \{x\}$ implies that $\exists m \in \mathbb{N}, m \leq n : a(1) = m$ and $a(n) = m + x$. Adding the constraints imposed by $t_{n-2}(a)$, we end up with 2 possible cases:

- $(a(1), a(2), a(n-1), a(n)) = (m, m+x-y, m+z, m+x)$; or
- $(a(1), a(2), a(n-1), a(n)) = (m, m+x-z, m+y, m+x)$

If $x = y$ or $x = z$, a cannot be a bijection, as it assumes a certain value twice; also, if $x = y + z$, it follows that $a(2) = a(n-1)$, in which case a cannot be a permutation unless $n - 1 = 2 \Leftrightarrow n = 3$. □

We will use this lemma repeatedly below. As a shorthand, we will denote the class of functions a satisfying the assumptions of the lemma as $[x|y, z]$.

4.1 Arrays of equal order

This case has already been studied in [5], in the context of electrical engineering: the authors remarked that "to increase the main lobe/sidelobe ratio [of a signal] without increasing the number of frequencies[. . .] it may be possible to stagger Costas signal pulses whose ambiguity sidelobe patterns do not coincide", and then proceeded to show this is not possible. For reasons of completeness we reproduce it here in full, albeit in a more compact and simplified form. Note, however, that this case alone is not sufficient to preclude "staggering" (what we call interlacing) as a strategy for building new Costas arrays; the case presented in the following section (Section 4.2) must also be studied.

Theorem 1. *Let a and b be Costas permutations of order $n > 3$, $n \in \mathbb{N}$, and let $T(a)$ and $T(b)$ be their difference triangles; then, $\exists i \in \{1, \ldots, n-1\} : t_i(a) \cap t_i(b) \neq \emptyset$, and, therefore, a and b are not orthogonal.*

Proof. We will offer a proof by contradiction. Assume that the statement is false, and consider the collection of multisets $T(a, b) = \{t_i(a, b) : i = 1, \ldots, n-1\}$ where $t_i(a, b) = t_i(a) \cup t_i(b)$, $i = 1, \ldots, n-1$: negating the conclusion of the theorem implies that these multisets are actually sets, since we are effectively assuming that $\forall i \in \{1, \ldots, n-1\} : t_i(a) \cap t_i(b) = \emptyset$; naturally, the very definition of the Costas property implies that all $t_i(a), t_i(b)$, $i = 1, \ldots, n-1$ are themselves sets and not multisets, since they cannot contain duplicate entries.

$T(a)$ must contain $n - 1$ entries of absolute value 1, and so must $T(b)$: consequently, $T(a, b)$ contains $2n - 2$ such values distributed over $n - 1$ rows. Now, no row can contain 3 or more such values, as then there would be 2 with the same sign, hence equal; and since $2n-2 = 2(n-1)$, each $t_i(a, b)$, $i = 1, \ldots, n-1$ must contain exactly 2 of them, necessarily of opposite sign.

The same argument can be repeated with the entries of absolute value 2. $T(a)$ must contain $n - 2$ entries of absolute value 2, and so must $T(b)$: consequently, $T(a, b)$ contains $2n - 4$ such values distributed over $n - 2$ rows. Again, no row can contain 3 or more such values, as then there would be 2 with the same sign, hence equal; and since $2n - 4 = 2(n - 2)$, each $t_i(a, b)$, $i = 1, \ldots, n-2$ must contain exactly 2 of them, necessarily of opposite sign.

Since $t_{n-1}(a, b)$ only has 2 entries, necessarily $t_{n-1}(a, b) = \{-1, 1\}$; similarly, since $t_{n-2}(a, b)$ only has 4 entries, necessarily $t_{n-2}(a, b) = \{-2, -1, 1, 2\}$, and, in the same way, we find that $t_{n-3}(a, b) = \{-3, -2, -1, 1, 2, 3\}$. We now need to determine which entries belong to $T(a)$, and whether the Costas property holds for each case:

- $[1| - 2, -1]$: Lemma 1 is of no help here; however, the 2 possibilities are $(a(1), a(2), a(n-1), a(n)) = (m, m+2, m-2, m+1)$ or $(m, m+3, m-1, m+1)$, and in both cases $t_{n-3}(a)$ should contain a -4, which cannot be the case, as $t_{n-3}(a, b)$ does not.
- $[1| - 2, 1]$: This is case $x = z$ in Lemma 1.
- $[1|-2, 2]$: Lemma 1 is of no help directly here. However, b should be a $[-1| - 1, 1]$ in this case, which is case $x = y$ in Lemma 1.
- $[1| - 1, 1]$: This is case $x = z$ in Lemma 1.
- $[1| - 1, 2]$: This is case $x = y + z$ in Lemma 1, so a cannot be Costas except perhaps when $n = 3$, in which case direct verification proves it is.
- $[1|1, 2]$: This is case $x = y$ in Lemma 1.

The remaining cases are also impossible, as they correspond to horizontal flips of the ones above. So, for $n > 3$, in all cases we proved it is impossible to construct a pair of orthogonal Costas arrays of equal order. □

4.2 Arrays of orders differing by 1

Theorem 2. *Let a and b be Costas permutations of orders n and $n+1$, respectively, with $n \in \mathbb{N}$, $n > 2$, and let $T(a)$ and $T(b)$ be their difference triangles; then, $\exists i \in \{1, \ldots, n-1\} : t_i(a) \cap t_i(b) \neq \emptyset$, and, therefore, a and b are not orthogonal.*

Proof. We proceed to offer a proof by contradiction along the same lines as in Theorem 1. Assuming that the statement is false, and considering the collection of multisets $T(a, b) = \{t_i(a, b) : i = 1, \ldots, n\}$ where $t_i(a, b) = t_i(a) \cup t_i(b)$, $i = 1, \ldots, n-1$, $t_n(a, b) = t_n(b)$, the negation of the conclusion of the theorem implies that these multisets are actually sets, since we are actually assuming that $\forall i \in \{1, \ldots, n-1\} : t_i(a) \cap t_i(b) = \emptyset$; note that the definition of the Costas property implies that all $t_i(a), t_i(b)$, $i = 1, \ldots, n-1$ are themselves sets and not multisets, since they cannot contain duplicate entries.

$T(a)$ must contain $n-1$ entries of absolute value 1, while $T(b)$ must contain n such values: consequently, $T(a, b)$ contains $2n-1$ such values distributed over n rows. Now, no row can contain 3 or more such values, as then there would be 2 with the same sign, hence equal; and since $2n-1 = 2(n-1)+1$, each $t_i(a, b)$, $i = 1, \ldots, n-1$ must contain exactly 2 of them, necessarily of opposite sign, while the remaining one will be $t_n(a, b)$'s only value: $t_n(a, b) = t_n(b) = \{\pm 1\}$.

The same argument can be repeated with the entries of absolute value 2. $T(a)$ must contain $n-2$ entries of absolute value 2, while $T(b)$ must contain $n-1$ such values: consequently, $T(a, b)$ contains $2n-3$ such values distributed over $n-1$ rows. Again, no row can contain 3 or more such values, as then there would be 2 with the same sign, hence equal; and since $2n-3 = 2(n-2)+1$, each $t_i(a, b)$, $i = 1, \ldots, n-2$ must contain exactly 2 of them, necessarily of opposite sign, while the remaining one will be in $t_{n-1}(a, b)$, which has 3 elements: $t_{n-1}(a, b) = \{-1, 1, \pm 2\}$. Repeating the argument verbatim with absolute value 3 yields that $t_{n-2}(a, b) = \{-2, -1, 1, 2, \pm 3\}$.

Let us now attempt to construct b explicitly based on that information; without loss of generality, let us assume that $t_n(a, b) = t_n(b) = \{1\}$ (otherwise flip both a and b horizontally). The following cases are possible:

- $[1|-1, 1]$: This is case $x = z$ in Lemma 1.

- $[1|-2, -1]$: Lemma 1 is of no help here; however, the 2 possibilities are $(b(1), b(2), b(n), b(n+1)) = (m, m+2, m-2, m+1)$ or $(m, m+3, m-1, m+1)$, and in both cases $t_{n-2}(a)$ should contain a -4, which cannot be the case, as $t_{n-2}(a, b)$ does not.

- $[1|-2, 1]$: This is case $x = z$ in Lemma 1.

- $[1|-1, 2]$: This is case $x = y + z$ in Lemma 1, so b cannot be Costas except perhaps when $n = 2$, in which case direct verification proves it is.

- $[1|1, 2]$: This is case $x = y$ in Lemma 1.

So, for $n > 2$, in all cases we proved it is impossible to construct a pair of orthogonal Costas arrays of orders differing by 1. Note that the cases appearing in this theorem also appear in Theorem 1, where they were proved not to lead to Costas permutations; noting this would make the proof shorter, but not self contained. □

4.3 Possible exceptions

Theorems 1 and 2 allow for exceptional interlaced constructions of Costas arrays in small orders $n \leq 6$. Do such arrays actually exist? Clearly Costas arrays of order 2 are trivially interlaced (as 2 Costas arrays of order 1). We checked exhaustively all Costas arrays of order $3 \leq n \leq 6$ to settle completely the issue for those orders where the theorems do not readily apply: none of these arrays was found to be interlaced. Together with the 2 aforementioned theorems, then, this fact proves

Theorem 3. *Interlaced Costas arrays of order $n \neq 2$ do not exist.*

5 Summary and conclusion

We have proved that interlaced Costas arrays cannot possibly exist, unless perhaps for very small orders, because any 2 Costas arrays of orders larger than 3 and differing at most by 1 must have a common distance vector, and interlacing dilates vectors by a factor of 2 but does not distort them. Further exhaustive search actually shows that the only interlaced Costas arrays are those of order 2. Therefore, though interlacing may initially seem a neat trick and an interesting idea for an empirical construction method, it is doomed to fail for all orders of interest. The result is a direct consequence of the Costas property, hence valid for all Costas arrays, and this is a rare case of a rigorous result that encompasses the whole family of Costas arrays, as opposed to those usually published that are valid only for (sub-families of) algebraically constructed Costas arrays.

As a possible future direction of research, we suggest investigating whether it is true that any 2 Costas arrays of arbitrary orders have a common vector and/or find conditions under which this occurs.

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