Higher dimensional generalizations of the Costas property

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***Abstract*—We investigate the generalization of the Costas property in 3 or more dimensions, and we seek an appropriate definition. We offer a construction method based on the idea of reshaping Costas arrays into higher-dimensional entities.**

I. Introduction

Costas arrays originated in engineering as a frequency hopping pattern that optimizes the performance of radars and sonars [3], [4]; being a singular combinatorial object, however, they have been lately the focus of intensive study by mathematicians as well, and have thus started leading a second independent "life" in the mathematical literature [7], [8], [9], [10], [14]. In this work, we increase the level of mathematical abstraction by investigating analogs of Costas arrays in 3 or more dimensions: several challenges lie ahead, as we first need to give a satisfactory definition of the Costas property in higher dimensions, and also provide some algorithms to construct such Costas cubes or "hypercubes" in general.

We will begin by stating the definition of Costas arrays [5] in such a way as to exhibit their direct relation to Golomb rulers [16], which will consequently be construed as the 1-dimensional analog of Costas arrays. Subsequently, we will discuss definitions in higher dimensions, and we will attempt to generalize the generation methods available to us.

II. The definition of Costas property in 1 and 2 dimensions

We use the usual notation that $[n]:=\{1,2,\ldots,n\},\ n\in\mathbb{N}$, and that $[n]-1:=\{0,1,\ldots,n-1\},\ n\in\mathbb{N}^*$.

Definition 1 (Costas property in 1 dimension)**.** Let $g:[N]\to[N'],\ N,N'\in\mathbb{N}$, be a finite strictly increasing sequence, such that $g(1)=1,\ g(N)=N'$; g will satisfy the *Costas property in one dimension* iff $g(i)-g(i+k)=g(j)-g(j+k)\Leftrightarrow i=j,\ i,j,i+k,j+k\in[N]$. Equivalently, if $f:[N']\to\{0,1\}$ is a sequence such that $f(i)=1\Leftrightarrow\exists j\in[N]:i=g(j)$, and $f(i)=0$ whenever $i\notin[N']$, then the *autocorrelation* $A_f(k)=\sum_{i\in[N']}f(i)f(i+k),\ k\in\mathbb{Z}$ satisfies $A_f(k)\leq 1,\ k\neq 0$. The sequence g is known a *Golomb ruler* [16].

Definition 2 (Costas property in 2 dimensions)**.** Let $f:\mathbb{Z}^2\to\{0,1\}$ be a binary sequence in 2 dimensions, such that it is equal to 1 at a finite number of points only; without loss of generality we may assume that $f(i,j)=0$ when $i\notin[M]$ or $j\notin[N]$ for some $M,N\in\mathbb{N}$. Defining the *autocorrelation* as $A_f(k,l)=\sum_{i,j\in\mathbb{Z}}f(i,j)f(i+k,j+l),\ k,l\in\mathbb{Z}$, f will have the *Costas property in 2 dimensions* iff $A_f(k,l)\leq 1,\ k,l\neq 0$.

Definition 3 (Costas squares)**.** Let f have the Costas property in 2 dimensions and let M,N, appearing in Definition 2 be the least possible; if $M=N=n$, f will be called a *Costas square* of side length n.

Definition 4 (Costas arrays)**.** Let f be a Costas square, and let it, in addition, have the structure of a permutation matrix with only one element equal to 1 per row and column: in other words, let there be a bijective sequence $g:[n]\to[n]$ (namely a *permutation* of order n) such that $f(i,j)=1\Leftrightarrow i=g(j),i,j\in[n]$. In that case, f will be called a *Costas array*, and g a *Costas permutation*.

Remark 1.

- It follows immediately that, if g is a Costas permutation of order n, no two vectors in the collection $\{(i-j,g(i)-g(j)):\ 1\leq i<j\leq n\}$ can be equal, and that no vector in this collection has a coordinate equal to 0.
- It is customary to denote 0s in the array f by blanks and 1s by dots.
- Note that the Costas property, as stated in Definition 2, is (much) more general than usual, as it can be satisfied by (many) more 2-dimensional sequences than Costas arrays themselves, which appear only as quite restricted special cases (see Definition 4). But this generality will actually prove to be helpful when we seek a higher-dimensional analog of Costas arrays.

III. Costas hypercubes

It is straightforward to generalize the Costas property in higher dimensions, to the extent that we risk (re)stating the obvious below. What is harder is to find the exact higher-dimensional analog of the special case, the "Costas array".

A. The Costas property in higher dimensions

Definition 5 (Costas hyper-rectangles and hypercubes)**.** Let $m\in\mathbb{N}$, consider a sequence $f:\mathbb{Z}^m\to\{0,1\}$, and suppose further that $f(i)=0,\ i\notin[N],N\in\mathbb{N}^m$, where $i=(i_1,\ldots,i_m)$, $N=(N_1,\ldots,N_m)$, $[N]=[N_1]\times\ldots\times[N_m]$, and the vector N has the smallest possible entries (for the given sequence f). Let the autocorrelation of f be

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$A_f(k) = \sum_{i \in \mathbb{Z}^m} f(i)f(i+k),\ k \in \mathbb{Z}^m$. Then, f will be a *Costas hyper-rectangle* iff $\forall k \in \mathbb{Z}^m - \{0\},\ A_f(k) \leq 1$. If f is a Costas hyper-rectangle so that $N_1 = \ldots = N_m = n \in \mathbb{N}$, it is called a *Costas hypercube*; if $m = 2$, it is a Costas square (see Definition 3), whereas if $m = 3$ it will be called a *Costas cube*.

Now it is time to seek the higher-dimensional analog of a permutation.

B. Vector permutations

It is quite simple to generalize permutations in higher dimensions, if the number of dimensions is even:

Definition 6 (Permutation Costas hypercube)**.** Let $m = 2s,\ s \in \mathbb{N}$, and let $g : [n]^s \to [n]^s$ be a bijection, that is a permutation on vectors in general. Let $f : \mathbb{Z}^m \to \{0, 1\}$ be a sequence such that $f(i) = 1$ iff $(i_{s+1}, \ldots, i_{2s}) = g(i_1, \ldots, i_s),\ (i_1, \ldots, i_s) \in [n]^s$, and such that it has the Costas property, as defined in Definition 5; then, f will be called a *permutation Costas hypercube* in m dimensions with side length n.

Remark 2.

- Permutation Costas hyper-rectangles are defined by the obvious extension of the above definition.
- The fact that we chose the first s dimensions to form the domain of g and the last s its range does not affect generality: if f is a Costas hypercube, then any f' resulting by a random permutation of the order of the dimensions is also a Costas hypercube; this constitutes a generalization of the invariance of the Costas property in 2 dimensions under transposition.
- No 2 vectors in the collection $\{(i - j, g(i) - g(j)) : i, j \in [n]^s, i \neq j\}$ can be equal; however, they may have coordinates equal to 0.

Example 1. As a specific example, let us use $m = 4$, $n = 3$. Then, the hypercube with $f(i) = 1$ iff i is one of the row vectors of Table III-B is a permutation Costas hypercube; this can be checked by a) verifying that the 2 first columns contain all vectors with integer coordinates between 1 and 3, as do the 2 last columns, and b) by finding all possible $\binom{9}{2} = 36$ distance vectors and observing they are indeed distinct. Observe also that this hypercube has in total $9 = 3^2$ nonzero elements out of $81 = 3^4$; in general, permutation Costas hypercubes have n^s nonzero elements out of n^{2s}.

Remark 3. When $m = 2s+1,\ s \in \mathbb{N}$, it is clearly impossible to define a permutation as we did above; the best we can aim for is to find an injective function $g : [n]^s \to n^{s+1}$, so that $f(i) = 1$ iff $(i_{s+1}, \ldots, i_{2s+1}) = g(i_1, \ldots, i_s),\ (i_1, \ldots, i_s) \in [n]^s$.

IV. The main construction method: reshaping

It is natural to ask whether Costas arrays (in 2 dimensions), which can be easily constructed or looked up in databases, such as the one by J.K. Beard [2], can somehow be manipulated (essentially reshaped) to produce Costas hypercubes, and hopefully permutation Costas hypercubes; this is indeed possible.

1	1	2	1
1	2	2	3
1	3	3	1
2	1	2	3
2	2	1	2
2	3	1	3
3	1	3	2
3	2	3	1
3	3	1	2

TABLE I
A hypercube with $m = 3$, $n = 4$ constructed by permuting all pairs of integers

A. Reshaping

We formulate and prove below a general result about constructing a Costas hyper-rectangle out of a Costas square. In the special case where the Costas square is a Costas array, and the hyper-rectangle a hypercube, it turns out the hypercube is a permutation Costas hypercube.

Theorem 1 (Reshaping)**.** *Let $m, n \in \mathbb{N}^*$, $n = \prod_{i=1}^{m} n_i$, $n_i > 1, i \in [m]$, and let g be a Costas permutation of order n, but following the convention that $g : [n] - 1 \to [n] - 1$. Expand $i = \sum_{j=1}^{m} v_j(i) \prod_{l=j+1}^{m} n_l$, so that i gets mapped bijectively to $V(i) = (v_1(i), \ldots, v_m(i))$, where $v_j \in [n_j] - 1,\ j \in [m]$; similarly, $g(i)$ gets mapped bijectively to $V(g(i)) = (v_1(g(i)), \ldots, v_m(g(i)))$. Then, the hyper-rectangle of side length n_i in dimension i and $i+m$, $i \in [m]$, whose dots (n in total) lie at the points $(V(i), V(g(i))) := (v_1(i), \ldots, v_m(i), v_1(g(i)), \ldots, v_m(g(i))),\ i \in [n] - 1$, is actually a permutation Costas hyper-rectangle.*

Proof: Choose 2 values for i, say i_1 and i_2; the corresponding distance vector is:

$$(V(i_1) - V(i_2), V(g(i_1)) - V(g(i_2))) = \\ = (v_1(i_1) - v_1(i_2), \ldots, v_m(i_1) - v_m(i_2), \\ v_1(g(i_1)) - v_1(g(i_2)), \ldots, v_m(g(i_1)) - v_m(g(i_2)))$$

We need to show that all of these vectors are distinct. In other words, we need to show:

$$(V(i_1) - V(i_2), V(g(i_1)) - V(g(i_2))) = \\ = (V(i_3) - V(i_4), V(g(i_3)) - V(g(i_4))) \Rightarrow i_1 = i_2,\ i_3 = i_4$$

Extend V^{-1} by $V^{-1}(v_1, \ldots, v_m) = \sum_{j=1}^{m} v_j \prod_{l=j+1}^{m} n_l$ on the class of vectors where $|v_j| < n_j,\ j \in [m]$. It follows that $V^{-1}(V(i_1) - V(i_2)) = i_1 - i_2$, as $V(i_1) - V(i_2)$ falls within this class of vectors. Putting things together:

$$(V(i_1) - V(i_2), V(g(i_1)) - V(g(i_2))) = \\ = (V(i_3) - V(i_4), V(g(i_3)) - V(g(i_4))) \Rightarrow$$

$$
\begin{aligned}
&V^{-1}(V(i_1)-V(i_2),V(g(i_1))-V(g(i_2)))= \\
&\quad =V^{-1}(V(i_3)-V(i_4),V(g(i_3))-V(g(i_4)))\Leftrightarrow \\
&(V^{-1}(V(i_1)-V(i_2)),V^{-1}(V(g(i_1))-V(g(i_2))))= \\
&=(V^{-1}(V(i_3)-V(i_4)),V^{-1}(V(g(i_3))-V(g(i_4))))\Leftrightarrow \\
&(i_1-i_2,g(i_1)-g(i_2))=(i_3-i_4,g(i_3)-g(i_4))\Leftrightarrow \\
&i_1-i_2=i_3-i_4,\ g(i_1)-g(i_2)=g(i_3)-g(i_4)\Rightarrow \\
&\qquad i_1=i_3,\ i_2=i_4
\end{aligned}
$$

In the last step we used the fact that g is Costas.

Further, observe that the left half row vectors so constructed are the expansions of $i \in [n]-1$, while the right half vectors are the expansions of $g(i),\ i \in [n]-1$; the fact that every $i \in [n]-1$ gets expanded exactly once and that g is a permutation guarantees that the hyper-rectangle produced is a permutation one. This completes the proof.

The hypercube of even dimension is just a special case:

Corollary 1 (Costas hypercube of even dimension)**.** Let $m, n \in \mathbb{N}^*$, and let g be a Costas permutation of order n^m, but following the convention that $g : [n]^m - 1 \to [n]^m - 1$. Expand $i = \sum_{j=1}^{m} v_j(i) n^{m-j}$, so that i gets mapped bijectively to $V(i) = (v_1(i), \ldots, v_m(i))$, where $v_j \in [n]-1,\ j \in [m]$; similarly, $g(i)$ gets mapped bijectively to $V(g(i)) = (v_1(g(i)), \ldots, v_m(g(i)))$. Then, the hypercube of side length n whose dots (n^m in total) lie at the points $(V(i), V(g(i))) := (v_1(i), \ldots, v_m(i), v_1(g(i)), \ldots, v_m(g(i))),\ i \in [n^m]-1$, is actually a permutation Costas hypercube.

Remark 4. Notice that these hypercubes have n^m dots, which is the square root of the n^{2m} total positions available in the hypercube, just like in Costas arrays, where there are n dots among the n^2 available positions.

Here is an attempt to construct *approximate* Costas hypercubes of odd dimension in the special case where the side length n is a perfect square, using Theorem 1, by first constructing a hyper-rectangle as an intermediate step:

Heuristic 1 (Costas hypercube of odd dimension)**.** *Let $m, n \in \mathbb{N}^*$, where $\sqrt{n} \in \mathbb{N}$, and let g be a Costas permutation of order $n^m\sqrt{n}$, but following the convention that $g : [n]^m \times [\sqrt{n}] - 1 \to [n]^m \times [\sqrt{n}] - 1$. Expand $i = v_0(i) n^m + \sum_{j=1}^{m} v_j(i) n^{m-j}$, so that i gets mapped bijectively to $V(i) = (v_0(i), v_1(i), \ldots, v_m(i))$, where $v_j \in [n]-1,\ j \in [m]$ and $v_0 \in [\sqrt{n}]-1$; similarly, $g(i)$ gets mapped bijectively to $V(g(i)) = (v_0(g(i)), v_1(g(i)), \ldots, v_m(g(i)))$. This process forms a Costas hyper-rectangle in $2m+2$ dimensions, whose side length in $2m$ dimensions is n and in the remaining 2 dimensions $\sqrt{n}$. Now, replace the coordinate pair $(v_0(i), v_0(g(i)))$ in the coordinate vector of each dot by the single coordinate $\sqrt{n} v_0(g(i)) + v_0(i)$, $i \in [n^m\sqrt{n}] - 1$, which takes values in the range $[n]-1$. Then, the hypercube of side length n whose dots ($n^m\sqrt{n}$ in total) lie at the points $(\sqrt{n} v_0(g(i)) + v_0(i), v_1(i), \ldots, v_m(i), v_1(g(i)), \ldots, v_m(g(i)))$, $i \in [n^m\sqrt{n}] - 1$, is usually a good approximation of a Costas hypercube, and the removal of a few dots turns it into a Costas hypercube.*

Remark 5. The reason why this heuristic often fails to produce a Costas hypercube is that different pairs of coordinates (in difference vectors) can collapse to the same value: for example, assume $n = 25 \Leftrightarrow \sqrt{n} = 5$ and consider the pairs $(-3, 0)$ and $(2, -1)$; they get mapped to $-3 + 5 \cdot 0 = -3$ and $2 - 5 = -3$. In the context of the proof of Theorem 1, this is equivalent to saying that $V(i_1) - V(i_2)$ is *not* always the same as $\text{sign}(i_1 - i_2) V(|i_1 - i_2|)$. However, simulations show that the damage this does to the Costas property is usually small: tests with Costas arrays of side length $n^3 \leq 200$ showed that systematically over 95% of the difference vectors among the dots are distinct.

The construction methods above can be significantly extended if we use Costas squares instead of Costas arrays as the starting point. The key observation (through the proof of Theorem 1) is that the permutation property of the original Costas array is not responsible for the Costas property of the hyper-rectangle produced, but rather for its permutation property alone (and in the case of Heuristic 1 it does not even achieve that). Therefore, if we are not interested in obtaining a permutation Costas hyper-rectangle as the final product, or if a suitable sized Costas array is not available, we may as well start with a Costas square.

A special type of Costas squares that proves very helpful in practice is smaller Costas arrays. Consider a Costas array of order $n' \in \mathbb{N}^*$ and let $n > n'$; then, this Costas array can be turned, by the addition of $n - n'$ blank rows and columns at the sides of the array, into a Costas square of size n. Note that such Costas squares are generated by incomplete permutations. We generalize this notion in the following definition:

Definition 7 (Incomplete Costas array)**.** A Costas square with the property that there is at most one dot per row and column will be called an *incomplete Costas array*. A (Costas) hyper-rectangle/hypercube of even dimension with the property that there are no 2 dots whose position vectors have the same left half or right half part will be called an *incomplete (Costas) hyper-rectangle/hypercube*.

Corollary 2 (Constructions out of Costas squares)**.**

- A construction according to Theorem 1 starting with a (incomplete) Costas square results to a (incomplete) Costas hyper-rectangle.
- A construction according to Corollary 1 starting with a (incomplete) Costas square results to a (incomplete) Costas hypercube.
- A construction according to Heuristic 1 starting with a (incomplete) Costas square results to a hypercube that very nearly has the Costas property and usually can be turned into a Costas hypercube through the removal of a few dots.

0	10
1	7
2	6
3	9
4	17
5	23
6	21
7	2
8	20
9	11
10	15
11	3
12	12
13	22
14	1
15	16
16	18
17	19
18	5
19	0
20	13
21	24
22	14
23	8
24	4

0	0	0	2
1	0	2	1
2	0	1	1
3	0	4	1
4	0	2	3
0	1	3	4
1	1	1	4
2	1	2	0
3	1	0	4
4	1	1	2
0	2	0	3
1	2	3	0
2	2	2	2
3	2	2	4
4	2	1	0
0	3	1	3
1	3	3	3
2	3	4	3
3	3	0	1
4	3	0	0
0	4	3	2
1	4	4	4
2	4	4	2
3	4	3	1
4	4	4	0

TABLE II
THE CONVERSION OF A COSTAS ARRAY OF ORDER 25 INTO A COSTAS HYPERCUBE WITH $m=4$, $n=5$: THE PERMUTATION (LEFT), AND THE FINAL COSTAS HYPERCUBE (RIGHT)

B. Construction examples

We give 3 examples: the first is the construction of a Costas hypercube with $n=5$, $m=4$ out of a Costas array of order 25 (using Corollary 1); the second is the construction of a Costas hypercube with $n=9$, $m=3$ out of a Costas array of order 27 (using Heuristic 1); and the third is the construction of an incomplete Costas hypercube with $m=5$, $n=4$ (using Heuristic 1), starting with a Costas array of order 31 and extending it into an incomplete Costas array of order 32.

Example 2. Consider the permutation of order 25 appearing on Table II (left). Applying Corollary 1, we get a Costas hypercube with $m=4$, $n=5$, whose dot positions appear on Table II (right). Observe that this is a permutation Costas hypercube, as Corollary 1 states: every vector $(i,j),\ i,j\in[5]-1$ appears in the left 2 columns in exactly 1 row and in the right 2 columns also in exactly one row.

Example 3. Consider the Costas permutation of order 27 appearing on Table III (left). Applying Heuristic 1, we first get a Costas hyper-rectangle in 4 dimensions of side lengths 9 and 3, whose dots lie at the points shown in Table III (center). Subsequently, we combine the middle columns that have values in the range $\{0,1,2\}$ into a single column with values in the range $[9]-1$, as described in Heuristic 1: for example, the middle 2 coordinates $(1,2)$ in row 12 become $1+2\cdot 3=7$. The result appears in Table III (right). A check of the Costas property shows that this time we get lucky and that the hypercube of $m=3$, $n=9$ we have created is Costas. It is obviously not a permutation Costas hypercube, as this term is meaningless in odd dimensions.

Example 4. Consider the Costas permutation of order 31 appearing on Table IV (left), and consider it as an incomplete Costas permutation of order $32=2^5=4^2\sqrt{4}$. Applying Heuristic 1, we first get a Costas hyper-rectangle in 6 dimen-

0	0
1	2
2	18
3	11
4	22
5	4
6	24
7	19
8	9
9	15
10	12
11	26
12	10
13	14
14	1
15	8
16	13
17	7
18	20
19	21
20	17
21	16
22	25
23	5
24	3
25	6
26	23

0	0	0	0
1	0	0	2
2	0	2	0
3	0	1	2
4	0	2	4
5	0	0	4
6	0	2	6
7	0	2	1
8	0	1	0
0	1	1	6
1	1	1	3
2	1	2	8
3	1	1	1
4	1	1	5
5	1	0	1
6	1	0	8
7	1	1	4
8	1	0	7
0	2	2	2
1	2	2	3
2	2	1	8
3	2	1	7
4	2	2	7
5	2	0	5
6	2	0	3
7	2	0	6
8	2	2	5

0	0	0
1	0	2
2	6	0
3	3	2
4	6	4
5	0	4
6	6	6
7	6	1
8	3	0
0	4	6
1	4	3
2	7	8
3	4	1
4	4	5
5	1	1
6	1	8
7	4	4
8	1	7
0	8	2
1	8	3
2	5	8
3	5	7
4	8	7
5	2	5
6	2	3
7	2	6
8	8	5

TABLE III
THE CONVERSION OF A COSTAS ARRAY OF ORDER 27 INTO A COSTAS HYPERCUBE WITH $m=3$, $n=9$: THE PERMUTATION (LEFT), THE INTERMEDIATE HYPER-RECTANGLE (CENTER), AND THE FINAL COSTAS HYPERCUBE (RIGHT)

sions of side lengths 4 and 2, whose dots lie at the points shown in Table IV (center). Subsequently, we combine the middle columns that have values in the range $\{0,1\}$ into a single column with values in the range $[4]-1$, as described in Heuristic 1. The result appears in Table IV (right). A check of the Costas property shows that this time we get lucky and that the hypercube of $m=5$, $n=4$ we have created is Costas. It is obviously not a permutation Costas hypercube, as this term is meaningless in odd dimensions. It also has just less than $\sqrt{4^5}=32$ dots, one less than that to be exact.

V. AN EXTENSION OF THE WELCH CONSTRUCTION

A. The original method and its non-extendability for Costas arrays

The Welch construction method for Costas arrays [5], [7] stipulates that, if p is a prime, g a primitive root [1] of $\mathbb{F}(p)$ and $c\in[p-1]-1$ a fixed parameter, then the function $f(i)=g^{i-1+c} \bmod p$, $i\in[p-1]$ is actually a Costas permutation on $[p-1]$. Contrary to the Golomb construction [5], [7], though, which works in all finite fields, namely with p^m elements where p is a prime and $m\in\mathbb{N}^*$, the Welch method is not applicable when $m>1$, as the elements of the field are then no longer represented by integers, but rather by polynomials of degree $m-1$, whose variable, say x, represents an algebraic element of $\mathbb{F}(p)$ of order m [1]), and with coefficients in $[p]-1$, while addition and multiplication are no longer defined modulo an integer, but rather modulo a monic irreducible polynomial $P(x)$ of degree m [1], [5]. It follows that the function f of the Welch construction is now $f:[q-1]\rightarrow\mathbb{F}^*(q)$ where $f(i)=g^{i-1+c} \bmod P(x),\ i\in[q-1]$, with $c\in[q-1]-1$ and g a primitive root of $\mathbb{F}(q)$; therefore, $f(i)$ is a polynomial, while i is an integer, and the whole construction is (at first

0	0
1	28
2	22
3	29
4	16
5	18
6	23
7	26
8	10
9	30
10	12
11	21
12	17
13	9
14	20
15	19
16	4
17	5
18	24
19	2
20	6
21	27
22	15
23	25
24	11
25	8
26	3
27	1
28	14
29	7
30	13

0	0	0	0	0	0
1	0	0	1	3	0
2	0	0	1	1	2
3	0	0	1	3	1
0	1	0	1	0	0
1	1	0	1	0	2
2	1	0	1	1	3
3	1	0	1	2	2
0	2	0	0	2	2
1	2	0	1	3	2
2	2	0	0	3	0
3	2	0	1	1	1
0	3	0	1	0	1
1	3	0	0	2	1
2	3	0	1	1	0
3	3	0	1	0	3
0	0	1	0	1	0
1	0	1	0	1	1
2	0	1	1	2	0
3	0	1	0	0	2
0	1	1	0	1	2
1	1	1	1	2	3
2	1	1	0	3	3
3	1	1	1	2	1
0	2	1	0	2	3
1	2	1	0	2	0
2	2	1	0	0	3
3	2	1	0	0	1
0	3	1	0	3	2
1	3	1	0	1	3
2	3	1	0	3	1

0	0	0	0	0
1	0	2	3	0
2	0	2	1	2
3	0	2	3	1
0	1	2	0	0
1	1	2	0	2
2	1	2	1	3
3	1	2	2	2
0	2	0	2	2
1	2	2	3	2
2	2	0	3	0
3	2	2	1	1
0	3	2	0	1
1	3	0	2	1
2	3	2	1	0
3	3	2	0	3
0	0	1	1	0
1	0	1	1	1
2	0	3	2	0
3	0	1	0	2
0	1	1	1	2
1	1	3	2	3
2	1	1	3	3
3	1	3	2	1
0	2	1	2	3
1	2	1	2	0
2	2	1	0	3
3	2	1	0	1
0	3	1	3	2
1	3	1	1	3
2	3	1	3	1

TABLE IV

THE CONVERSION OF A COSTAS ARRAY OF ORDER 31 INTO A COSTAS HYPERCUBE WITH $m = 5$, $n = 4$, TREATING THE COSTAS ARRAY AS AN INCOMPLETE COSTAS ARRAY OF ORDER 32: THE (INCOMPLETE) PERMUTATION (LEFT), THE INTERMEDIATE HYPER-RECTANGLE (CENTER), AND THE FINAL COSTAS HYPERCUBE (RIGHT)

sight, at least) meaningless, as we need f to produce integer values!

Although this construction fails to yield a Costas array, it still produces Costas hyper-rectangles, as we are about to see.

B. Construction of hyper-rectangles

The field $\mathbb{F}(p^m)$ can be construed to be a vector space over the field $\mathbb{F}(p)$ in 2 ways, depending on whether we consider its elements to be polynomials or m-tuples:

Definition 8. The field $\mathbb{F}(p^m)$, where p prime and $m \in \mathbb{N}^*$, when viewed as a vector space over the field $\mathbb{F}(p)$, will be denoted by $\mathbb{F}(p)^m$. Let $V_P(p,m)$ denote the vector space of polynomials of degree $m-1$ over the field $\mathbb{F}(p)$. Then, $V_P(p,m)$ and $\mathbb{F}(p)^m$ are isomorphic vector spaces, as they have the same finite dimension m and they are over the same field [1], under the isomorphism denoted by $\mathcal{F}$, whereby $\mathcal{F}(a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \ldots + a_0) = (a_{m-1}, a_{m-2}, \ldots, a_0)$, $a_i \in \mathbb{F}(p)$, $i \in [m]-1$.

Theorem 2 (Welch hyper-rectangles and hypercubes)**.** *Let p be a prime, $m \in \mathbb{N}^*$, g a primitive root of $\mathbb{F}(q)$ where $q = p^m$, and $c \in [q-1]-1$. We choose $V_P(p,m)$ as our representation of $\mathbb{F}(q)$. Then:*

- *The function $f : [q-1] \to \mathbb{F}^*(q)$, where $f(i) = \mathcal{F}\left(g^{i-1+c} \bmod P(x)\right)$, $i \in [q-1]$, $P(x)$ an irreducible polynomial over $\mathbb{F}(p)$ of degree m, is a permutation over $\mathbb{F}^*(q)$.*
- *The hyper-rectangle in $m+1$ dimensions with side length $q-1$ in the first dimension and p in the others, whose dots lie at the positions with coordinates $\{(i, f(i))|i \in [q-1]\}$, has the Costas property.*
- *The hypercube in $2m$ dimensions with side length p, whose dots lie at the positions with coordinates $\{(V(i), f(i))|i \in [q-1]\}$, has the Costas property; V is the familiar mapping from Corollary 1, with $n = p$ in the present case.*

Proof: The proof really consists of putting together bits we have already proved.

- f is a permutation over $\mathbb{F}^*(q)$ because g is taken to be a primitive root of $\mathbb{F}(q)$.
- That the family of vectors $\{(i, f(i))|i \in [q-1]\}$ has the Costas property follows from a verbatim repetition of the classical argument for $m = 2$ [5], [7].
- We need to show that, given that the family $\{(i, f(i))|i \in [q-1]\}$ has the Costas property, the family $\{(V(i), f(i))|i \in [q-1]\}$ has it too; but this is a verbatim repetition of the argument presented in the proof of Theorem 1.

Remark 6.

- The hypercubes constructed above have $q-1$ dots out of q^2 possible dot positions, and thus follow approximately the square root rule we saw earlier for the density.
- They are "almost" permutation hypercubes, except that the zero vectors are missing. If we set $f(0) = 0$ and add a dot at the position $(V(0), f(0)) = (0, 0)$ (obviously $V(0) = 0$ as well), then we get a hypercube but we have no guarantee anymore that it has the Costas property; simulations show that sometimes it does. This is the equivalent of the W_0 construction of Costas arrays by the addition of a "corner dot" to a W_1-constructed array [5], [7], [8].
- Welch arrays retain the Costas property when their columns get shifted circularly [5], [7]; this shift is expressed by the fixed parameter c in the definition of the permutation f shown earlier. As the extension of the Welch method formulated in Theorem 2 preserves this parameter c in the definition of f, we see that the hypercubes and the hyper-rectangles it produces have a similar periodicity: the fact that the family of dots at the locations $\{(i, f(i))|i \in [q-1]\}$ defines a Costas hyper-rectangle implies that the family of dots at the locations $\{(i, f(i \oplus k))|i \in [q-1]\}$, where $i \oplus k := 1 + [(i+k-1) \bmod (q-1)]$, $k \in [q-1]-1$ fixed, also defines a Costas hyper-rectangle; similarly, the fact that the family of dots at the locations $\{(V(i), f(i))|i \in [q-1]\}$ defines a Costas hypercube implies that the family of dots at the locations $\{(V(i), f(i \oplus k))|i \in [q-1]\}$, $k \in [q-1]-1$ fixed, also defines a Costas hypercube.
- The most important aspect of this method is that it builds hypercubes not derived by Costas squares or arrays, at least in an obvious way. At the risk of sounding overly optimistic, if a method that converts Costas hypercubes into Costas arrays were available, it could potentially lead to novel Costas arrays when applied on these hypercubes.

1	0	1	0
2	1	0	0
3	0	1	2
4	1	2	0
5	2	1	2
6	1	1	1
7	1	2	2
8	2	0	2
9	0	1	1
10	1	1	0
11	1	1	2
12	1	0	2
13	0	0	2
14	0	2	0
15	2	0	0
16	0	2	1
17	2	1	0
18	1	2	1
19	2	2	2
20	2	1	1
21	1	0	1
22	0	2	2
23	2	2	0
24	2	2	1
25	2	0	1
26	0	0	1

0	0	1	0	1	0
0	0	2	1	0	0
0	1	0	0	1	2
0	1	1	1	2	0
0	1	2	2	1	2
0	2	0	1	1	1
0	2	1	1	2	2
0	2	2	2	0	2
1	0	0	0	1	1
1	0	1	1	1	0
1	0	2	1	1	2
1	1	0	1	0	2
1	1	1	0	0	2
1	1	2	0	2	0
1	2	0	2	0	0
1	2	1	0	2	1
1	2	2	2	1	0
2	0	0	1	2	1
2	0	1	2	2	2
2	0	2	2	1	1
2	1	0	1	0	1
2	1	1	0	2	2
2	1	2	2	2	0
2	2	0	2	2	1
2	2	1	2	0	1
2	2	2	0	0	1

1	3
2	1
3	21
4	7
5	23
6	13
7	25
8	20
9	12
10	4
11	22
12	19
13	18
14	6
15	2
16	15
17	5
18	16
19	26
20	14
21	10
22	24
23	8
24	17
25	11
26	9

TABLE V
THE PROCESS OF CONSTRUCTING A WELCH HYPERCUBE IN $\mathbb{F}(27)$: THE WELCH HYPER-RECTANGLE CORRESPONDING TO $g = x$, $c = 0$, $P(x) = x^3 + 2x + 1$ (LEFT), THE CORRESPONDING WELCH HYPERCUBE (CENTER), AND THE CORRESPONDING WELCH PERMUTATION (RIGHT)

Example 5. Let $p = 3$ and $m = 3$, so that $q = 27$, choose $P(x) = x^3 + 2x + 1$ which is irreducible over $\mathbb{F}(3)$, and choose $c = 1$, $g = x$. The Costas hyper-rectangle and the Costas hypercube constructed by Theorem 2 are shown in Table V, along with the corresponding Welch permutation discussed in Remark 6, which fails to have the Costas property. In this particular example, adding a corner dot at $(0, 0)$ to the hypercube preserves the Costas property, thus yielding a permutation Costas hypercube.

VI. SUMMARY, CONCLUSION, AND FUTURE DIRECTIONS

We defined the multidimensional generalization of Costas arrays in several possible ways, by investigating what the multidimensional generalization of the Costas property should be. We adopted the point of view that the Costas property of a multidimensional binary sequence depends exclusively on its autocorrelation, and that the permutation structure (which we also suitably defined in higher dimensions) is just an extra condition imposed, not directly related to the Costas property itself.

Hence, Costas arrays became special cases of sequences with the Costas property in 2 dimensions, which we named Costas rectangles and squares; while the former generalized naturally to Costas hyper-rectangles and hypercubes, we generalized the latter as permutation Costas hypercubes, which were viewed as the representation of a permutation between vectors instead of integers.

For the case of permutation Costas hyper-rectangles and hypercubes of even dimension, we proposed a construction method that reshapes an existing Costas array of suitable order into the desired hyper-rectangle or hypercube. In the case of odd dimension, we proposed a related heuristic that does not always work, but even when it doesn't it usually produces a hyper-rectangle that very nearly has the Costas property. We also generalized the constructions by starting with Costas squares instead of arrays, and gave specific examples.

We subsequently investigated the application of the Welch construction method on finite fields with a nonprime number of elements, and found out that, although the method fails to produce Costas arrays, it produces Costas hyper-rectangles and hypercubes in a natural way. Once more, we supplied specific examples of the construction.

There are still many possible directions for future research in Costas hypercubes. For example,

1) Heuristic 1 seems to be producing Costas hypercubes pretty often, although in many occasions it fails to do so. Can we provide a rigorous and simple sufficient condition for the resulting hypercube to have the Costas property?
2) Is there a different construction method for Costas hypercubes? In particular, can the Welch and Golomb methods be generalized in a direct way (that is, without the intermediate step of the use of the mapping V defines in Theorem 1) in 3 or more dimensions? More generally, can a construction method be found directly based on finite fields?
3) Are there any engineering applications of Costas hypercubes, perhaps of a similar nature to the applications of Costas arrays?
4) Is there a method to convert a Costas hypercube into a Costas array? Such a method could potentially lead to the construction of new Costas arrays, because some of the construction methods we have proposed, such as the extended Welch method in Section V, produce (permutation) Costas hypercubes not linked to any Costas array, at least in an obvious way.

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